On Singular Value Decomposition in Clifford Algebras

Dmitry Shirokov

HSE University, Moscow, Russia

International Conference "Modern computational technologies for mathematical modeling"
dedicated to the 75th anniversary of Professor V. I. Vasil'ev,
June 4–9, 2024

This work is supported by the Russian Science Foundation (project 23-71-10028).

Clifford geometric algebra (GA)

Let us consider the real Clifford geometric algebra (GA) $\mathcal{G}_{p,q}$ with the identity element $e \equiv 1$ and the generators e_a, $a = 1, 2, \dots, n$, where $n = p + q \geq 1$:

$$e_a e_b + e_b e_a = 2\eta_{ab} e, \qquad \eta = (\eta_{ab}) = \mathrm{diag}(1, \dots, 1, -1, \dots, -1).$$

Consider the subspaces $\mathcal{G}^k_{p,q}$ of grades $k = 0, 1, \dots, n$, which elements are linear combinations of the basis elements $e_A = e_{a_1 a_2 \dots a_k} = e_{a_1} e_{a_2} \cdots e_{a_k}$ with ordered multi-indices of length k. An arbitrary multivector $M \in \mathcal{G}_{p,q}$ has the form

$$M = \sum_A m_A e_A \in \mathcal{G}_{p,q}, \qquad m_A \in \mathbb{R},$$

where we have a sum over arbitrary multi-index A of length from 0 to n. The projection of M onto the subspace $\mathcal{G}^k_{p,q}$ is denoted by $\langle M \rangle_k$.

The grade involution and reversion of a multivector $M \in \mathcal{G}_{p,q}$ are denoted by

$$\widehat{M} = \sum_{k=0}^{n} (-1)^k \langle M \rangle_k, \qquad \widetilde{M} = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \langle M \rangle_k \tag{1}$$

$$\widehat{M_1 M_2} = \widehat{M_1} \widehat{M_2}, \qquad \widetilde{M_1 M_2} = \widetilde{M_2} \widetilde{M_1}, \qquad \forall M_1, M_2 \in \mathcal{G}_{p,q}. \tag{2}$$

Euclidean space on GA

Let us consider an operation of Hermitian conjugation $\dagger$ in $\mathcal{G}_{p,q}$:

$$M^\dagger := M|_{e_A \to (e_A)^{-1}} = \sum_A m_A (e_A)^{-1}. \tag{3}$$

We have the following two other equivalent definitions of this operation:

$$M^\dagger = \begin{cases} e_{1\ldots p}\widetilde{M} e^{-1}_{1\ldots p}, & \text{if } p \text{ is odd,} \\ e_{1\ldots p}\widehat{\widetilde{M}} e^{-1}_{1\ldots p}, & \text{if } p \text{ is even,} \end{cases} = \begin{cases} e_{p+1\ldots n}\widetilde{M} e^{-1}_{p+1\ldots n}, & \text{if } q \text{ is even,} \\ e_{p+1\ldots n}\widehat{\widetilde{M}} e^{-1}_{p+1\ldots n}, & \text{if } q \text{ is odd.} \end{cases} \tag{4}$$

The operation

$$(M_1, M_2) := \langle M_1^\dagger M_2 \rangle_0 \geq 0$$

is a (positive definite) scalar product. Using this scalar product we introduce inner product space over the field of real numbers (euclidean space) in $\mathcal{G}_{p,q}$. We have a norm

$$||M|| := \sqrt{(M, M)} = \sqrt{\langle M^\dagger M \rangle_0} \tag{5}$$

Matrix representation of $\mathcal{G}_{p,q}$

Let us consider the following faithful representation (isomorphism) of the real geometric algebra $\mathcal{G}_{p,q}$

$$\beta:\mathcal{G}_{p,q} \to \begin{cases} \mathrm{Mat}(2^{\frac{n}{2}},\mathbb{R}), & \text{if } p-q=0,2 \mod 8,\\ \mathrm{Mat}(2^{\frac{n-1}{2}},\mathbb{R})\oplus\mathrm{Mat}(2^{\frac{n-1}{2}},\mathbb{R}), & \text{if } p-q=1 \mod 8,\\ \mathrm{Mat}(2^{\frac{n-1}{2}},\mathbb{C}), & \text{if } p-q=3,7 \mod 8,\\ \mathrm{Mat}(2^{\frac{n-2}{2}},\mathbb{H}), & \text{if } p-q=4,6 \mod 8,\\ \mathrm{Mat}(2^{\frac{n-3}{2}},\mathbb{H})\oplus\mathrm{Mat}(2^{\frac{n-3}{2}},\mathbb{H}), & \text{if } p-q=5 \mod 8. \end{cases} \quad (6)$$

These isomorphisms are known as Cartan–Bott 8-periodicity.
Let us denote the size of the corresponding matrices by

$$d:=\begin{cases} 2^{\frac{n}{2}}, & \text{if } p-q=0,2 \mod 8,\\ 2^{\frac{n+1}{2}}, & \text{if } p-q=1 \mod 8,\\ 2^{\frac{n-1}{2}}, & \text{if } p-q=3,5,7 \mod 8,\\ 2^{\frac{n-2}{2}}, & \text{if } p-q=4,6 \mod 8. \end{cases} \quad (7)$$

Note that we use block-diagonal matrices in the cases $p-q=1,5 \mod 8$.

Let us present an explicit form of one β' of these representations of $\mathcal{G}_{p,q}$. We have $\beta'(e) = I$ and $\beta'(e_{a_1 a_2 \ldots a_k}) = \beta'(e_{a_1})\beta'(e_{a_2}) \cdots \beta'(e_{a_k})$.
In some particular cases, we construct β' in the following way:

- In the case $\mathcal{G}_{0,1}$: $e_1 \to i$.
- In the case $\mathcal{G}_{1,0}$: $e_1 \to \mathrm{diag}(1, -1)$.
- In the case $\mathcal{G}_{0,2}$: $e_1 \to i$, $e_2 \to j$.
- In the case $\mathcal{G}_{0,3}$: $e_1 \to \mathrm{diag}(i, -i)$, $e_2 \to \mathrm{diag}(j, -j)$, $e_3 \to \mathrm{diag}(k, -k)$.

Suppose we know $\beta'_a := \beta'(e_a)$, $a = 1, \ldots, n$ for some fixed $\mathcal{G}_{p,q}$, $p + q = n$. Then we construct explicit matrix representation of $\mathcal{G}_{p+1,q+1}$, $\mathcal{G}_{q+1,p-1}$, $\mathcal{G}_{p-4,q-4}$ in the following way using the matrices β'_a, $a = 1, \ldots, n$.

- In the case $\mathcal{G}_{p+1,q+1}$: $e_a \to \mathrm{diag}(\beta'_a, -\beta'_a)$, $a = 1, \ldots, p, p+2, \ldots, p+q+1$.
  In the subcase $p - q \neq 1 \mod 4$, we have

$$e_{p+1} \to \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad e_{p+q+2} \to \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

  In the subcase $p - q = 1 \mod 4$, we have

$$e_{p+1} \to \mathrm{diag}(\beta_1 \cdots \beta_n \Omega, -\beta_1 \cdots \beta_n \Omega),\ e_{p+q+2} \to \mathrm{diag}(\Omega, -\Omega),\ \Omega = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

- In the case $\mathcal{G}_{q+1,p-1}$: $e_1 \to \beta'_1$, $e_i \to \beta'_i \beta'_1$, $i = 2, \ldots, n$.
- In the case $\mathcal{G}_{p-4,q+4}$: $e_i \to \beta'_i \beta'_1 \beta'_2 \beta'_3 \beta'_4$, $i = 1, 2, 3, 4$, $e_j \to \beta'_j$, $j = 5, \ldots, n$.

It can be directly verified that for this matrix representation we have

$$\eta_{aa}\beta'(e_a) = \begin{cases} (\beta'(e_a))^{\mathrm{T}}, & \text{if } p-q = 0,1,2 \mod 8, \\ (\beta'(e_a))^{\mathrm{H}}, & \text{if } p-q = 3,7 \mod 8, \\ (\beta'(e_a))^{*}, & \text{if } p-q = 4,5,6 \mod 8, \end{cases} \qquad a = 1,\ldots,n, \tag{8}$$

where T is transpose of a (real) matrix, H is the Hermitian transpose of a (complex) matrix, $*$ is the conjugate transpose of a matrix over quaternions. Using the linearity, we get that these matrix conjugations are consistent with Hermitian conjugation of corresponding multivector:

$$\beta'(M^{\dagger}) = \begin{cases} (\beta'(M))^{\mathrm{T}}, & \text{if } p-q = 0,1,2 \mod 8, \\ (\beta'(M))^{\mathrm{H}}, & \text{if } p-q = 3,7 \mod 8, \\ (\beta'(M))^{*}, & \text{if } p-q = 4,5,6 \mod 8, \end{cases} \qquad M \in \mathcal{G}_{p,q}. \tag{9}$$

Note that the formulas like (9) are not valid for an arbitrary matrix representation β of the form (6). They are true for the matrix representations $\gamma = T^{-1}\beta' T$ obtained from β' by the matrix T such that

- $T^{\mathrm{T}} T = I$ in the cases $p-q = 0,1,2 \mod 8$,
- $T^{\mathrm{H}} T = I$ in the cases $p-q = 3,7 \mod 8$,
- $T^{*} T = I$ in the cases $p-q = 4,5,6 \mod 8$.

Lie groups

Let us consider the following Lie group in $\mathcal{G}_{p,q}$

$$\mathrm{G}\mathcal{G}_{p,q} = \{M \in \mathcal{G}_{p,q} : M^\dagger M = e\}. \tag{10}$$

Note that all the basis elements e_A of $\mathcal{G}_{p,q}$ belong to this group by the definition. Using (6) and (9), we get the following isomorphisms of this group to the classical matrix Lie groups:

$$\mathrm{G}\mathcal{G}_{p,q} \simeq \begin{cases} \mathrm{O}(2^{\frac{n}{2}}), & \text{if } p-q = 0,2 \mod 8,\\ \mathrm{O}(2^{\frac{n-1}{2}}) \times \mathrm{O}(2^{\frac{n-1}{2}}), & \text{if } p-q = 1 \mod 8,\\ \mathrm{U}(2^{\frac{n-1}{2}}), & \text{if } p-q = 3,7 \mod 8,\\ \mathrm{Sp}(2^{\frac{n-2}{2}}), & \text{if } p-q = 4,6 \mod 8,\\ \mathrm{Sp}(2^{\frac{n-3}{2}}) \times \mathrm{Sp}(2^{\frac{n-3}{2}}), & \text{if } p-q = 5 \mod 8, \end{cases} \tag{11}$$

where we have the following notation for (orthogonal, unitary, and simplectic correspondingly) classical matrix Lie groups

$$\mathrm{O}(k) = \{A \in \mathrm{Mat}(k,\mathbb{R}) : \quad A^{\mathrm{T}} A = I\}, \tag{12}$$

$$\mathrm{U}(k) = \{A \in \mathrm{Mat}(k,\mathbb{C}) : \quad A^{\mathrm{H}} A = I\}, \tag{13}$$

$$\mathrm{Sp}(k) = \{A \in \mathrm{Mat}(k,\mathbb{H}) : \quad A^* A = I\}. \tag{14}$$

Singular value decomposition (SVD)

Theorem

For an arbitrary $A \in \mathbb{R}^{n\times m}$, there exist matrices $U \in \mathrm{O}(n)$ and $V \in \mathrm{O}(m)$ such that

$$A = U\Sigma V^{\mathrm{T}}, \qquad (15)$$

where

$$\Sigma = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k), \qquad k = \min(n, m), \qquad \mathbb{R} \ni \lambda_1, \lambda_2, \ldots, \lambda_k \geq 0.$$

Note that choosing matrices $U \in \mathrm{O}(n)$ and $V \in \mathrm{O}(m)$, we can always arrange diagonal elements of the matrix Σ in decreasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$.

Diagonal elements of the matrix Σ are called singular values, they are square roots of eigenvalues of the matrices AA^{T} or $A^{\mathrm{T}}A$. Columns of the matrices U and V are eigenvectors of thematrices AA^{T} and $A^{\mathrm{T}}A$ respectively.

Theorem

For an arbitrary $A \in \mathbb{C}^{n\times m}$*, there exist matrices* $U \in \mathrm{U}(n)$ *and* $V \in \mathrm{U}(m)$ *such that* $A = U\Sigma V^{\mathrm{H}}$*, where*

$$\Sigma = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k), \qquad k = \min(n, m), \qquad \mathbb{R} \ni \lambda_1, \lambda_2, \ldots, \lambda_k \geq 0.$$

Note that choosing matrices $U \in \mathrm{U}(n)$ *and* $V \in \mathrm{U}(m)$*, we can always arrange diagonal elements of the matrix* Σ *in decreasing order* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$*.*

Diagonal elements of the matrix Σ are called singular values, they are square roots of eigenvalues of the matrices AA^{H} or $A^{\mathrm{H}}A$. Columns of the matrices U and V are eigenvectors of the matrices AA^{H} and $A^{\mathrm{H}}A$ respectively.

Theorem

For an arbitrary $A \in \mathbb{H}^{n\times m}$*, there exist matrices* $U \in \mathrm{Sp}(n)$ *and* $V \in \mathrm{Sp}(m)$ *such that* $A = U\Sigma V^{*}$*, where*

$$\Sigma = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k), \qquad k = \min(n, m), \qquad \mathbb{R} \ni \lambda_1, \lambda_2, \ldots, \lambda_k \geq 0.$$

Diagonal elements of the matrix Σ are called singular values.

Theorem (SVD in GA)

For an arbitrary multivector $M \in \mathcal{G}_{p,q}$*, there exist multivectors* $U, V \in \mathrm{G}\mathcal{G}_{p,q}$*, where*

$$\mathrm{G}\mathcal{G}_{p,q} = \{U \in \mathcal{G}_{p,q} : U^\dagger U = e\}, \quad U^\dagger := \sum_A u_A (e_A)^{-1},$$

such that

$$M = U \Sigma V^\dagger, \tag{16}$$

where multivector Σ *belongs to the subset* K *of* $\mathcal{G}_{p,q}$*, which is real span of a set of* d *fixed basis elements (always including the identity element* e*):*

$$\Sigma \in K := \mathrm{span}(\{e_{B_i}, i = 1, \ldots, d\}) = \{\sum_{i=1}^{d} \lambda_i e_{B_i}, \quad \lambda_i \in \mathbb{R}\}, \tag{17}$$

$$d := \begin{cases} 2^{\frac{n}{2}}, & \text{if } p-q = 0,2 \mod 8, \\ 2^{\frac{n+1}{2}}, & \text{if } p-q = 1 \mod 8, \\ 2^{\frac{n-1}{2}}, & \text{if } p-q = 3,5,7 \mod 8, \\ 2^{\frac{n-2}{2}}, & \text{if } p-q = 4,6 \mod 8. \end{cases} \tag{18}$$

Thus the meaning of SVD in geometric algebra is the following:

after multiplication on the left and on the right by elements of the group $\mathrm{G}\mathcal{G}_{p,q}$, any multivector $M \in \mathcal{G}_{p,q}$, $\dim \mathcal{G}_{p,q} = 2^n$, can be placed in a d-dimensional subspace K of $\mathcal{G}_{p,q}$, where d is

$$d := \begin{cases} 2^{\frac{n}{2}}, & \text{if } p-q = 0,2 \mod 8,\\ 2^{\frac{n+1}{2}}, & \text{if } p-q = 1 \mod 8,\\ 2^{\frac{n-1}{2}}, & \text{if } p-q = 3,5,7 \mod 8,\\ 2^{\frac{n-2}{2}}, & \text{if } p-q = 4,6 \mod 8. \end{cases}$$

Example

In the case $\mathcal{G}_{2,0}$, we have

$$\beta'(e)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \beta'(e_1)=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \beta'(e_2)=\begin{pmatrix}-1&0\\0&1\end{pmatrix},\ \beta'(e_{12})=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

The matrices $\beta'(e)$ and $\beta'(e_2)$ are real and diagonal, we get the 2-dimensional subspace

$$K=\mathrm{span}(e,e_2).$$

Example

In the case $\mathcal{G}_{2,1}$, the matrices $\beta'(e)$, $\beta'(e_1)$, $\beta'(e_{23})$, and $\beta'(e_{123})$ are real and diagonal. We get the 4-dimensional subspace

$$K=\mathrm{span}(e,e_1,e_{23},e_{123}).$$

Example

In the case $\mathcal{G}_{1,3}$, the matrices $\beta'(e)$, $\beta'(e_{14})$ are real and diagonal. We get the 2-dimensional subspace

$$K=\mathrm{span}(e,e_{14}).$$

Theorem (Polar decomposition)

For an arbitrary $A \in \mathbb{R}^{n\times n}$*, there exist positive semi-definite symmetric matrices* P *and* $S \in \mathbb{R}^{n\times n}$ *(i.e.* $P^{\mathrm{T}} = P$ *and* $z^{\mathrm{T}}Pz \geq 0$, $\forall z \in \mathbb{R}^n$*;* $S^{\mathrm{T}} = S$ *and* $z^{\mathrm{T}}Sz \geq 0$, $\forall z \in \mathbb{R}^n$*) and matrix* $W \in \mathrm{O}(n)$ *such that*

$$A = WP = SW. \tag{19}$$

Given a real symmetric matrix P, the following statements are equivalent:

- P is positive semi-definite,
- all the eigenvalues of P are non-negative,
- there exists a matrix B such that $P = B^{\mathrm{T}}B$.

If we have SVD of the real matrix $A = U\Sigma V^{\mathrm{T}}$, then we can take $W = UV^{\mathrm{T}}$, $P = V\Sigma V^{\mathrm{T}}$, and $S = U\Sigma U^{\mathrm{T}}$. Note that $P = \sqrt{A^{\mathrm{T}}A}$ and $S = WPW^{\mathrm{T}} = \sqrt{AA^{\mathrm{T}}}$.

Theorem

For an arbitrary $A \in \mathbb{C}^{n\times n}$*, there exist positive semi-definite Hermitian matrices* P *and* $S \in \mathbb{C}^{n\times n}$ *(i.e.* $P^{\mathrm{H}} = P$ *and* $z^{\mathrm{H}}Pz \geq 0$, $\forall z \in \mathbb{C}^n$*;* $S^{\mathrm{H}} = S$ *and* $z^{\mathrm{H}}Sz \geq 0$, $\forall z \in \mathbb{C}^n$*) and matrix* $W \in \mathrm{U}(n)$ *such that*

$$A = WP = SW. \tag{20}$$

Given a complex Hermitian matrix P, the following statements are equivalent:

- P is positive semi-definite,
- all the eigenvalues of P are non-negative,
- there exists a matrix B such that $P = B^{\mathrm{H}}B$.

If we have SVD of the complex matrix $A = U\Sigma V^{\mathrm{H}}$, then we can take $W = UV^{\mathrm{H}}$, $P = V\Sigma V^{\mathrm{H}}$, and $S = U\Sigma U^{\mathrm{H}}$. Note that $P = \sqrt{A^{\mathrm{H}}A}$ and $S = WPW^{\mathrm{H}} = \sqrt{AA^{\mathrm{H}}}$.

Theorem

For an arbitrary $A \in \mathbb{H}^{n\times n}$, there exist quaternion positive semi-definite Hermitian matrices P and $S \in \mathbb{H}^{n\times n}$ (i.e. $P^* = P$ and $z^*Pz \geq 0$, $\forall z \in \mathbb{H}^n$; $S^* = S$ and $z^*Sz \geq 0$, $\forall z \in \mathbb{H}^n$) and matrix $W \in \mathrm{Sp}(n)$ such that

$$A = WP = SW. \tag{21}$$

Given a quaternion Hermitian matrix P, the following statements are equivalent:

- P is positive semi-definite,
- all the eigenvalues of P are non-negative,
- there exists a matrix B such that $P = B^*B$.

If we have SVD of the quaternion matrix $A = U\Sigma V^*$, then we can take $W = UV^*$, $P = V\Sigma V^*$, and $S = U\Sigma U^*$. Note that $P = \sqrt{A^*A}$ and $S = WPW^* = \sqrt{AA^*}$.

Theorem (Left and right polar decomposition in GA)

For an arbitrary multivector $M\in\mathcal{G}_{p,q}$, there exist multivectors $P,S\in\mathcal{G}_{p,q}$ such that

$$P^\dagger=P,\qquad S^\dagger=S,\qquad U^\dagger:=\sum_A u_A(e_A)^{-1}, \tag{22}$$

$$P=B^\dagger B,\qquad S=C^\dagger C\qquad \textit{for some multivectors } B,C\in\mathcal{G}_{p,q},, \tag{23}$$

and multivector

$$W\in \mathrm{G}\mathcal{G}_{p,q}=\{U\in\mathcal{G}_{p,q}:U^\dagger U=e\}$$

such that

$$M=WP=SW.$$

Note that

$$P=\sqrt{M^\dagger M},\qquad S=WPW^\dagger=\sqrt{MM^\dagger}. \tag{24}$$

If we have the SVD of multivector $M=U\Sigma V^\dagger$ (16), then

$$W=UV^\dagger,\qquad P=V\Sigma V^\dagger,\qquad S=U\Sigma U^\dagger. \tag{25}$$

Conclusions

- We naturally implement SVD and polar decomposition in GA without using the corresponding matrix representations. The new theorems involve only operations in geometric algebras. The polar decomposition is a consequence of the SVD.
- We use matrix representations in the proofs, namely, we use the classical SVD and polar decomposition of real, complex, and quaternion matrices. It could be interesting to investigate, in a future work, alternative and more direct proofs involving only operations in the corresponding GA.
- We do not present a method (algorithm) to find the SVD in GA. We present an existing theorem. How to find elements Σ, U, and V using only the methods of GA and without using the corresponding matrix representations is a good task for further research. The problems of numerical accuracy and computation speed can also be considered.
- We expect the use of the theorems in different applications of GA in computer science, engineering, physics, big data, machine learning, etc.

Shirokov D., *On Singular Value Decomposition and Polar Decomposition in Geometric Algebras*. Advances in Computer Graphics. CGI 2023. Lecture Notes in Computer Science, 14498, Springer, Cham, 2024, 391–401.

Shirokov D., *On SVD and Polar Decomposition in Real and Complexified Clifford Algebras*. Advances in Applied Clifford Algebras, 34 (2024), 23, 20 pp., arXiv:2404.11920.

Marchuk N. G., Shirokov D. S., *Theory of Clifford algebras and spinors* (in Russian), URSS, Moscow, 560 pp., 2020.

Shirokov D. S., *Lectures on Clifford algebras and spinors* (in Russian), Lects. Kursy NOC 19, Steklov Math. Inst., RAS, Moscow, 180 pp., 2012.

Marchuk N. G., Shirokov D. S., *Introduction to the theory of Clifford algebras* (in Russian), Phasis, Moscow, 590 pp., 2012

Shirokov D. S., *Clifford algebras and their applications to Lie groups and spinors* (Lectures), Proceedings of the Nineteenth International Conference on Geometry, Integrability and Quantization, Avangard Prima, Sofia, Bulgaria, 2018, arXiv:1709.06608.

Thank you for your attention!